

# CHARACTER SHEAVES ON UNIPOTENT GROUPS IN CHARACTERISTIC $p > 0$

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## Overview

These are slides for a talk given by the authors at the conference “Current developments and directions in the Langlands program” held in honor of Robert Langlands at the Northwestern University in May of 2008. The research program outlined in this talk was realized in a series of articles [1]–[4]. The orbit method for unipotent groups in positive characteristic is discussed in [1]. The results on character sheaves discussed in parts I and II of this talk are proved in [3]. The results described in part III are proved in [2] and [4]; the former studies  $L$ -packets of irreducible characters and the latter is devoted to the relationship between characters and character sheaves on unipotent groups over finite fields.

- [1] M. Boyarchenko and V. Drinfeld, *A motivated introduction to character sheaves and the orbit method for unipotent groups in positive characteristic*, Preprint, [math/0609769](#)
- [2] M. Boyarchenko, *Characters of unipotent groups over finite fields*, [arXiv:0712.2614](#), *Selecta Math.* **16** (2010), no. 4, 857–933.
- [3] M. Boyarchenko and V. Drinfeld, *Character sheaves on unipotent groups in positive characteristic: foundations*, [arXiv:0810.0794](#), to appear in *Selecta Mathematica*.
- [4] M. Boyarchenko, *Character sheaves and characters of unipotent groups over finite fields*, [arXiv:1006.2476](#), to appear in the *American Journal of Mathematics*.

## Some historical comments

A geometric approach to representation theory for unipotent groups: orbit method (Dixmier, Kirillov and others; late 1950s – early 1960s). Does not work in characteristic  $p > 0$  unless additional assumptions on the group are made.

The geometric approach to studying irreducible representations of groups of the form  $G(\mathbb{F}_q)$ , where  $G$  is a reductive group over  $\mathbb{F}_q$ , is at the heart of Deligne-Lusztig theory (1970s) and Lusztig's theory of character sheaves (1980s).

In 2003 Lusztig explained that there should also exist an interesting theory of character sheaves for unipotent groups in char.  $p > 0$ .

We will outline a theory that combines some of the essential features of Lusztig's theory and of the orbit method.

# Part I. Definition of character sheaves

Some notation

$X$  = a scheme of finite type over a field  $k$

$G$  = an algebraic group over  $k$

$\ell$  = a prime different from  $\text{char } k$

$\mathcal{D}(X) := D_c^b(X, \overline{\mathbb{Q}}_\ell)$ , the bounded derived category of constructible complexes of  $\overline{\mathbb{Q}}_\ell$ -sheaves on  $X$

Given a  $G$ -action on  $X$ , one can define  $\mathcal{D}_G(X)$ , the  $G$ -equivariant derived category.

In this talk we will mostly be concerned with  $\mathcal{D}_G(G)$ , defined on the next slide, where  $G$  acts on itself by conjugation.

## The category $\mathcal{D}_G(G)$

$G$  = a unipotent group over a field  $k$

$\ell$  = a prime different from  $\text{char } k$

$\mathcal{D}(G) = D_c^b(G, \overline{\mathbb{Q}}_\ell)$ , as before

$\mu : G \times G \longrightarrow G$  is the multiplication map

$c : G \times G \longrightarrow G$  is the conjugation action

$p_2 : G \times G \longrightarrow G$  is the second projection

$\mathcal{D}_G(G)$  consists of pairs  $(M, \phi)$ , where  $M \in \mathcal{D}(G)$  and  $\phi : c^*M \xrightarrow{\sim} p_2^*M$  satisfies the obvious cocycle condition

$\mathcal{D}(G)$  is monoidal, and  $\mathcal{D}_G(G)$  is braided monoidal, with respect to convolution (with compact supports):

$$M * N = R\mu_!(M \boxtimes N)$$

(Recall that a *braiding* is a certain type of a commutativity constraint.)

## Idempotents in monoidal categories

(1) A weak idempotent in a monoidal category  $(\mathcal{M}, \otimes, \mathbb{1})$  is an object  $e$  such that  $e \otimes e \cong e$ .

(2) A closed idempotent in  $\mathcal{M}$  is an object  $e$  such that there is an arrow  $\pi : \mathbb{1} \longrightarrow e$ , which becomes an isomorphism after applying  $e \otimes -$  as well as after applying  $- \otimes e$

It is more convenient to work with the second notion, because it is much more rigid (e.g.,  $\pi$  is unique up to a unique automorphism of  $e$ ).

In each of the two contexts, we have the notion of the Hecke subcategory  $e\mathcal{M}e$ . If  $e$  is closed,  $e\mathcal{M}e$  is monoidal as well, with unit object  $e$ .

When  $\mathcal{M}$  is additive and braided, we can also talk about minimal (weak/closed) idempotents.

## Motivation behind idempotents

1. **Classical fact:**  $\Gamma =$  finite group;

$\text{Fun}(\Gamma) =$  algebra of functions  $\Gamma \longrightarrow \overline{\mathbb{Q}}_\ell$  w.r.t. pointwise addition and convolution;

$\text{Fun}(\Gamma)^\Gamma =$  center of  $\text{Fun}(\Gamma)$

$\exists$  a bijection between the set  $\hat{\Gamma}$  of irreducible characters of  $\Gamma$  over  $\overline{\mathbb{Q}}_\ell$  and the set of minimal idempotents in  $\text{Fun}(\Gamma)^\Gamma$ : namely,  $\chi \longmapsto \frac{\chi(1)}{|\Gamma|} \cdot \chi$

2. **Explanation of the term “closed”:** Let  $X$  be a scheme of finite type over  $k$ , and let  $\mathcal{M} = \mathcal{D}(X)$ , with  $\otimes = \bigotimes_{\overline{\mathbb{Q}}_\ell}^L$ . Then the closed idempotents in  $\mathcal{M}$  are of the form  $i_! \overline{\mathbb{Q}}_\ell$ , where  $i : Y \hookrightarrow X$  is the inclusion of a *closed* subscheme and  $\overline{\mathbb{Q}}_\ell$  is the constant sheaf on  $Y$ .

*Typically (e.g., in this example) there are many weak idempotents that are not closed.*

3. **Orbit method:**  $k = \bar{k}$ ,  $\text{char } k = p > 0$ ;  
 $G =$  connected unipotent group over  $k$ ;  
 assume  $G$  has nilpotence class  $< p$ ;  
 form  $\mathfrak{g} = \text{Log } G$  and  $\mathfrak{g}^* =$  Serre dual of  $\mathfrak{g}$

Then  $G$  acts on  $\mathfrak{g}$  and  $\mathfrak{g}^*$ , and we can consider

$$\mathcal{D}_G(G) \xrightarrow{\exp^*} \mathcal{D}_G(\mathfrak{g}) \xrightarrow{\text{Fourier}} \mathcal{D}_G(\mathfrak{g}^*),$$

which yields a bijection between minimal weak idempotents in  $\mathcal{D}_G(G)$  and  $G$ -orbits in  $\mathfrak{g}^*$ .

Note that the  $G$ -orbits in  $\mathfrak{g}^*$  are all closed (as  $G$  is unipotent). Hence all minimal weak idempotents in  $\mathcal{D}_G(G)$  are closed as well.

In fact, the last statement remains true for any unipotent group  $G$  over  $k$ .



## Character sheaves and $L$ -packets

$k$  = an algebraically closed field of char.  $p > 0$

$G$  = an arbitrary unipotent group over  $k$

Pick a minimal closed idempotent  $e \in \mathcal{D}_G(G)$ .

Note that  $e\mathcal{D}_G(G)e = e\mathcal{D}_G(G)$ . Consider

$$\mathcal{M}_e^{perv} = \{(M, \phi) \in e\mathcal{D}_G(G) \mid M \text{ is perverse}\},$$

a full additive subcategory of  $\mathcal{D}_G(G)$ .

**Definition.** The  $L$ -packet of character sheaves associated to  $e$  is the set of indecomposable objects of  $\mathcal{M}_e^{perv}$ . An object of  $\mathcal{D}_G(G)$  is a character sheaf if it lies in some  $L$ -packet.

**Conjecture.** (1)  $\mathcal{M}_e^{perv}$  is a semisimple abelian category with finitely many simple objects.

(2)  $\exists n_e \in \mathbb{Z}$  such that  $e[-n_e] \in \mathcal{M}_e^{perv}$ .

(3)  $\mathcal{M}_e := \mathcal{M}_e^{perv}[n_e]$  is closed under  $*$ , and is a *modular* ( $\approx$ as far from being symmetric as possible) braided monoidal category.

# Part II. Construction of character sheaves

## Overview

**Two classical results.** (1) If  $\Gamma$  is a finite nilpotent group and  $\rho \in \hat{\Gamma}$ , there exist a subgroup  $H \subset \Gamma$  and a homomorphism  $\chi : H \rightarrow \overline{\mathbb{Q}}_\ell^\times$  such that  $\rho = \rho_{H,\chi} := \text{Ind}_H^\Gamma \chi$ .

(2) For *any* such pair  $(H, \chi)$ , Mackey's criterion states that  $\rho_{H,\chi}$  is irreducible  $\iff \forall \gamma \in \Gamma \setminus H$ ,

$$\chi|_{H \cap \gamma H \gamma^{-1}} \not\equiv \chi^\gamma|_{H \cap \gamma H \gamma^{-1}}$$

The notion of a 1-dimensional representation, the operation of induction, and the two results stated above, have geometric analogues. This is what the second part will be about.

## 1-dimensional character sheaves

A geometrization of the notion of a 1-dimensional representation is provided by the notion of a multiplicative local system.

$G$  = an algebraic group over a field  $k$

$\ell$  = a prime different from  $\text{char } k$

$\mu : G \times G \longrightarrow G$  is the multiplication map

A nonzero  $\overline{\mathbb{Q}}_\ell$ -local system  $\mathcal{L}$  on  $G$  is said to be multiplicative if  $\mu^* \mathcal{L} \cong \mathcal{L} \boxtimes \mathcal{L}$  (so  $\text{rk } \mathcal{L} = 1$ ).

If  $G$  is connected and unipotent, and  $\mathbb{K}_G$  is the dualizing complex of  $G$ , then  $e_{\mathcal{L}} := \mathcal{L} \otimes \mathbb{K}_G$  is a minimal closed idempotent in  $\mathcal{D}_G(G)$ .

If, moreover,  $k = \overline{k}$ , the corresponding  $L$ -packet consists of the single character sheaf  $\mathcal{L}[\dim G]$ .

## A more canonical viewpoint

Fix an embedding  $\psi : \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \overline{\mathbb{Q}}_\ell^\times$ . If  $\Gamma$  is a finite  $p$ -group, every homomorphism  $\Gamma \rightarrow \overline{\mathbb{Q}}_\ell^\times$  factors through  $\psi$ . Note that  $\text{Hom}(\Gamma, \mathbb{Q}_p/\mathbb{Z}_p)$  does not depend on  $\ell$ .

Now let  $G$  be an algebraic group over a field  $k$  of char.  $p > 0$ . We have the functors

$$\begin{aligned} & \left\{ \text{central extensions } 1 \rightarrow \mathbb{Q}_p/\mathbb{Z}_p \rightarrow \tilde{G} \rightarrow G \rightarrow 1 \right\} \\ & \xrightarrow{\text{forgetful}} \left\{ \text{multiplicative } \mathbb{Q}_p/\mathbb{Z}_p\text{-torsors on } G \right\} \\ & \xrightarrow{\psi_*} \left\{ \text{multiplicative } \overline{\mathbb{Q}}_\ell\text{-local systems on } G \right\} \end{aligned}$$

If  $G$  is *connected* and *unipotent*, they induce bijections on isomorphism classes of objects.

So we will study central extensions by  $\mathbb{Q}_p/\mathbb{Z}_p$  in place of multiplicative local systems.

The composition of the two functors will be denoted by  $\chi \longmapsto \mathcal{L}_\chi$ .

## Serre duality

$k$  = perfect field of char.  $p > 0$

$G$  = connected unipotent group over  $k$

The *Serre dual* of  $G$  is the functor

$$G^* : \left\{ \begin{array}{c} \text{perfect} \\ k\text{-schemes} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{abelian} \\ \text{groups} \end{array} \right\}$$

$$S \longmapsto \left\{ \text{iso. classes of central extensions} \right.$$

$$\left. \text{of the group scheme } G \times_k S \text{ by } \mathbb{Q}_p/\mathbb{Z}_p \times S \right\}$$

**Classical Serre duality.** If  $G$  is commutative, then  $G^*$  is representable by a perfect connected commutative unipotent group scheme over  $k$ .

**Proposition.** In general,  $G^*$  is representable by a possibly disconnected perfect commutative unipotent group over  $k$ . Its neutral connected component,  $(G^*)^\circ$ , can be naturally identified with  $(G^{ab})^*$ .

## Definition of admissible pairs

$k$  = algebraically closed field of char.  $p > 0$

$G$  = connected unipotent group over  $k$

An admissible pair for  $G$  is a pair  $(H, \chi)$ , where  $H \subset G$  is a connected subgroup and  $\chi \in H^*(k)$ , such that the following three conditions hold:

(1) Let  $G'$  be the normalizer in  $G$  of the pair  $(H, \chi)$ ; then  $G'^{\circ}/H$  is commutative.

(2) The homomorphism  $G'^{\circ}/H \rightarrow (G'^{\circ}/H)^*$  induced by  $\chi$  (a geometrization of  $g \mapsto \chi([g, -])$ ) is an isogeny (i.e., has finite kernel).

(3) If  $g \in G(k) \setminus G'(k)$ , the restrictions of  $\chi$  and  $\chi^g$  to  $(H \cap g^{-1}Hg)^{\circ}$  are nonisomorphic.

This is the correct geometric analogue of Mackey's irreducibility criterion.

## Induction with compact supports

$G =$  unipotent group over a field  $k$

$G' \subset G$  is a closed subgroup

One can define a functor

$$\mathrm{ind}_{G'}^G : \mathcal{D}_{G'}(G') \longrightarrow \mathcal{D}_G(G),$$

called induction with compact supports.

Its construction is standard: take  $M \in \mathcal{D}_{G'}(G')$ , extend by zero to  $\overline{M} \in \mathcal{D}(G)$ , then average, *in the sense of “lower shriek”*, with respect to the conjugation action of  $G$  on itself.

If  $k$  is finite *and*  $G'$  *is connected*, this functor is compatible with induction of functions via the sheaves-to-functions dictionary. (If  $G'$  is not connected, this may be false, in general.)

## Construction of $L$ -packets

$k$  = algebraically closed field of char.  $p > 0$

$G$  = connected unipotent group over  $k$

$(H, \chi)$  = an admissible pair for  $G$

$G'$  = normalizer of  $(H, \chi)$  in  $G$

Define:  $\mathcal{L}_\chi$  = multiplicative  $\overline{\mathbb{Q}_\ell}$ -local system on  $H$  arising from  $\chi$  via a chosen  $\psi : \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \overline{\mathbb{Q}_\ell}^\times$ ;

$e_\chi = \mathcal{L}_\chi \otimes \mathbb{K}_H \cong \mathcal{L}_\chi[2 \dim H] \in \mathcal{D}_{G'}(H)$ ;

$\mathcal{D}_{G'}(G') \ni e$  = extension of  $e_\chi$  by zero;

$\mathcal{D}_G(G) \ni f = \mathrm{ind}_{G'}^G e$

**Theorem.** (1)  $f$  is a minimal closed idempotent in  $\mathcal{D}_G(G)$ .

(2)  $\mathrm{ind}_{G'}^G$  restricts to a braided monoidal equivalence  $e\mathcal{D}_{G'}(G') \xrightarrow{\sim} f\mathcal{D}_G(G)$ .



(3) Set  $n_e = \dim H$ ,  $n_f = \dim H - \dim(G/G')$ ,

$$\mathcal{M}_e = \mathcal{M}_e^{perv}[n_e], \quad \mathcal{M}_f = \mathcal{M}_f^{perv}[n_f].$$

Then  $\mathcal{M}_e$  and  $\mathcal{M}_f$  are closed under  $*$ , and

$$\mathrm{ind}_{G'}^G(\mathcal{M}_e) = \mathcal{M}_f.$$

(4) The categories  $\mathcal{M}_e$  and  $\mathcal{M}_f$  are semisimple and have finitely many simple objects.

It follows that the  $L$ -packet of character sheaves on  $G$  associated to  $f$  is formed by the objects  $\mathrm{ind}_{G'}^G(M_j)[-n_f]$ , where  $M_1, \dots, M_k \in \mathcal{M}_e$  are the simple objects. The objects  $M_j$  can be described explicitly.

(5) Every minimal *weak* idempotent  $f \in \mathcal{D}_G(G)$  arises from an admissible pair  $(H, \chi)$  as above, and, in particular, is necessarily *closed*.

*However,  $(H, \chi)$  is usually far from unique.*

## Part III. Relation to characters

### $L$ -packets of irreducible characters

$\mathbb{F}_q$  = finite field with  $q$  elements

$G_0$  = connected unipotent group over  $\mathbb{F}_q$

$$G = G_0 \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$$

$\text{Fr} : G \rightarrow G$  is the Frobenius endomorphism

$\mathcal{P}$  = a  $\text{Fr}$ -stable  $L$ -packet of character sheaves on  $G$  (equivalently, the corresponding minimal idempotent  $e \in \mathcal{D}_G(G)$  satisfies  $\text{Fr}^* e \cong e$ )

Define

$$\mathcal{P}^{\text{Fr}} = \{N \in \mathcal{P} \mid \text{Fr}^*(N) \cong N\}.$$

For each  $M \in \mathcal{P}^{\text{Fr}}$ , form the corresponding function  $t_M : G_0(\mathbb{F}_q) \rightarrow \overline{\mathbb{Q}_\ell}$  (it is well defined up to rescaling).

We define a set  $\mathcal{P}'$  of irreducible characters of  $G_0(\mathbb{F}_q)$  as follows:  $\omega \in \mathcal{P}' \iff \omega$  lies in the span of the set of functions  $\{t_M\}_{M \in \mathcal{P}^{\text{Fr}}}$ .

We call  $\mathcal{P}' \subset \widehat{G_0(\mathbb{F}_q)}$  the  $L$ -packet of irreducible characters defined by the Fr-stable  $L$ -packet  $\mathcal{P}$  of character sheaves.

**Theorem.** Every irreducible character of  $G_0(\mathbb{F}_q)$  over  $\overline{\mathbb{Q}_\ell}$  lies in an  $L$ -packet as defined above.

There exists a description of  $L$ -packets of irreducible characters of  $G_0(\mathbb{F}_q)$  in terms of admissible pairs for  $G_0$ , which is analogous to the statement we discussed earlier, but is independent of the theory of character sheaves.

This description plays an important role in the proof of the last theorem.

## Description of $L$ -packets of characters

Let  $G_0$  be a connected unipotent group over  $\mathbb{F}_q$ , and consider all pairs  $(H, \chi)$  consisting of a connected subgroup  $H \subset G_0$  and an element  $\chi \in H^*(\mathbb{F}_q)$  such that  $(H \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \chi \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q)$  is an admissible pair for  $G = G_0 \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ .

Two such pairs,  $(H_1, \chi_1)$  and  $(H_2, \chi_2)$ , are geometrically conjugate if they are conjugate by an element of  $G(\overline{\mathbb{F}}_q)$ .

Let  $\mathcal{C}$  be a geometric conjugacy class of such pairs. For an irrep  $\rho$  of  $G_0(\mathbb{F}_q)$  over  $\overline{\mathbb{Q}}_\ell$ , we will write  $\rho \in \mathcal{P}_{\mathcal{C}}$  if there exists  $(H, \chi) \in \mathcal{C}$  such that  $\rho$  is an irreducible constituent of  $\text{Ind}_{H(\mathbb{F}_q)}^{G_0(\mathbb{F}_q)} t_{\mathcal{L}_\chi}$ .

**Theorem.** The  $\mathcal{P}_{\mathcal{C}}$  are exactly the  $L$ -packets of irreducible characters of  $G_0(\mathbb{F}_q)$ .

It could happen that  $\mathcal{C}_1 \neq \mathcal{C}_2$  and  $\mathcal{P}_{\mathcal{C}_1} = \mathcal{P}_{\mathcal{C}_2}$ .